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Quasi-Para-Sasakian Manifold Admitting Canonical Paracontact Connection

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Abstract

The purpose of the present paper is to study various geometric properties of Quasi-Para-Sasakian manifold with respect to canonical paracontact connection. A unique relation between curvature tensors of canonical paracontact connection and Levi-Civita connection have been obtained. We study quasi M-projectively flat as well as M-projectively flat quasi-para-Sasakian manifold with respect to canonical paracontact connection. Moreover, it is shown that a $\phi-$M-projectively flat quasi-para-Sasakian manifold with respect to canonical paracontact connection is an $\eta-$Eienstain Manifold. Also, we study quasi-para-Sasakian manifold with respect to canonical paracontact connection satisfying $\bar{M}(\xi, U).\bar{R} = 0$ and $\bar{M}(\xi, U).\bar{S} = 0$, where $\bar{M}$, $\bar{R}$ and $\bar{S}$ are M-Projective curvature tensor, curvature tensor and Ricci tensor with respect to canonical paracontact connection.

Keywords: Quasi-Para-Sasakian manifold; Projective curvature tensor; Canonical paracontact connection.

2010 Mathematics Subject Classification: 53C25, 53D15

1 Introduction

In [9] Kaneyuki and Konzai defined the almost paracontact structure on pseudo-Riemannian manifold M^n of dimension $(2n+1)$ and constructed the almost paracontact structure on $M^{(2n+1)} \times \Re$.

In 2008, S. Zamkovoy associated the almost paracontact structure [29] to a pseudo-Riemannian metric of signature $(n+1, n)$ and showed that any almost paracontact structure admits such a pseudo-Riemannian metric which is called compatible metric. He introduced the canonical paracontact connection and it showed that its torsion is the obstruction the paracontact manifold to be para-Sasakian. He defined a canonical paracontact connection on a paracontact metric manifold which seems to be the paracontact anologue of the (generalized) Tanaka-Webster connection [27].

In ([3], [2]), A. Biswas and K. K. Baishya introduced canonical paracontact connection for a generalized pseudo Ricci symmetric Sasakian manifolds as well as for an almost pseudo symmetric Sasakian manifolds. This affine connection was further studied by A. M. Blaga [4]. A. Mandal and A. Das ([12], [13], [14]). For an n-dimensional almost contact metric manifold M^n equipped with an almost contact

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metric structure (ϕ, ξ, η, g) consisting of a (1,1) tensor field ϕ, a vector field ξ, a 1-form η and a Riemannian metric g the canonical paracontact connection is defined by

$$\bar{\nabla}_X Y = \nabla_X Y + (\nabla_X \eta)(Y)\xi - \eta(Y)\nabla_X \xi + \eta(X)\phi Y. \tag{1.1}$$

A systematic study of almost paracontact metric manifold was given in one of the Zamkovoy's papers [29]. Z. Olszak [20] studied normal almost contact metric manifolds of dimension 3. He obtained certain necessary and sufficient conditions for an almost contact metric structure to be normal and curvature properties of such structures were studied. Normal almost paracontact metric manifolds were studied by many other authors ([1], [10], [11], [28]).

The notion of quasi-Sasakion manifolds was introduced by D. E. Blair [5]. A quasi-Sasakian manifold is a normal almost contact metric manifold whose fundamental 2-form $\phi := g(., \phi.)$ is closed. Quasi-Sasakian manifolds can be viwed as an odd-dimensional counterpart of Kaehler structures. These manifolds have been studied by many authors ([8], [19], [26]). Although quasi-Sasakian manifolds were studied by several different authors and are considered a well-established topic in contact Riemannian geometry to the authors knowledge, there do not exists any comprehensive study about quasi-para-Sasakian manifold.

In 1971, Pokhariyal and Mishra [21] defined a curvature tensor on Riemannian manifold as

$$\begin{aligned} M(X,Y)Z =& R(X,Y)Z - \frac{1}{2(n-1)}[S(Y,Z)X - S(X,Z)Y \\ &+ g(X,Z)QY - g(Y,Z)QX], \end{aligned} \tag{1.2}$$

which is known as M-projective curvature tensor, where R denotes the Riemannian curvature tensor of type (0,3), S denotes the Ricci tensor of type (0,2) and Q denotes the Ricci operator, for all $X,\ Y,\ Z \in \chi(M)$. In ([16], [17]), R. H. Ojha studied some curvature properties of M-projective curvature tensor in Sasakian manifolds and showed that a M-projective recurrent Sasakian manifold is M-projectively flat if and only if it is an Einstein manifold. M-projective curvature tensor was studies by several authors ([6], [7], [15], [18], [22], [23], [24]).

In a quasi-para-Sasakian manifold M^n of dimension $n > 2$, the M-projective curvature tensor $\bar{M}$ with respect to canonical paracontact connection $\bar{\nabla}$ is given by

$$\begin{aligned} \bar{M}(X,Y)Z =& \bar{R}(X,Y)Z - \frac{1}{2(n-1)}[\bar{S}(Y,Z)X - \bar{S}(X,Z)Y \\ &+ g(X,Z)\bar{Q}Y - g(Y,Z)\bar{Q}X], \end{aligned} \tag{1.3}$$

where $\bar{R},\ \bar{S}\ and\ \bar{Q}$ are Riemannian curvature tensor, Ricci tensor and Ricci operator with respect to canonical paracontact connection $\bar{\nabla}$ respectively.

Motivated by these considerations, in this paper, we studied the M-projective curvature tensor of quasi-para Sasakian manifold admitting canonical paracontact connection and established some new results.

Definition 1.1. *An n-dimensional quasi-para Sasakian manifold M^n is said to be $\eta-$Einstein manifold if the Ricci tensor S is of the form $S(X,Y) = ag(X,Y) + b\eta(X)\eta(Y)$, for all $X,\ Y \in \chi(M)$, where a and b are scalars.*

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2 Preliminaries

An odd dimensional smooth manifold M^n (n=2m+1) has an almost paracontact structure (ϕ, ξ, η) if it admits a tensor field ϕ of type (1, 1), a vector field ξ and a 1-form η satisfying the following conditions:

$$\phi(\xi) = 0, \tag{2.1}$$

$$\eta \circ \phi = 0, \tag{2.2}$$

$$\eta(\xi) = 1, \tag{2.3}$$

$$\phi^2 X = X - \eta(X)\xi. \tag{2.4}$$

Distribution $D : P \in M \to D_P \subseteq T_P M$; $D_P = Ker\eta = \{X \in T_P M : \eta(X) = 0\}$ is called paracontact distribution generated by η.

If a manifold M^n with (ϕ, ξ, η) structure admits a pseudo-Riemannian metric g such that

$$g(\phi X, \phi Y) = -g(X, Y) + \eta(X)\eta(Y), \tag{2.5}$$

then we say that M^n has an almost paracontact metric structure and g is called compatible. Any compatible metric g with the given almost paracontact structure g is necesserialy of signature $(n+1, n)$. Also if

$$\eta(X) = g(X, \xi) \tag{2.6}$$

and

$$g(X, \phi Y) = d\eta(X, Y), \tag{2.7}$$

where,

$$d\eta(X, Y) = \frac{1}{2}\{X\eta(Y) - Y\eta(X) - \eta[X, Y]\}, \tag{2.8}$$

holds then η is paracontact form and the almost paracontact metric manifold (M, ϕ, η, ξ, g) is said to be paracontact metric manifold.

A paracontact metric manifold is para-Sasakian manifold if and only if

$$(\nabla_X \varphi)(Y) = -g(X, Y)\xi + \eta(Y)X, \tag{2.9}$$

for all vector fields X and Y.

If

$$(\nabla_X \phi)(Y) = g(X, Y)\xi - \eta(Y)X, \tag{2.10}$$

then the manifold (M, ϕ, η, ξ, g) is said to be a quasi-para-Sasakian manifold.

Also

$$g(X, \phi Y) = -g(\phi X, Y), \tag{2.11}$$

$$(\nabla_X \eta)(Y) = -g(X, \phi Y), \tag{2.12}$$

$$(\nabla_X \xi) = \phi X, \tag{2.13}$$

$$d\eta(X, Y) = -g(X, \phi Y), \tag{2.14}$$

$$R(X, Y)\xi = \eta(X)Y - \eta(Y)X, \tag{2.15}$$

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$$R(X,\xi)Y = g(X,Y)\xi - \eta(Y)X, \tag{2.16}$$

$$S(\phi X,\phi Y) = -(n-1)g(\phi X,\phi Y), \tag{2.17}$$

$$S(X,\xi) = -(n-1)\eta(X). \tag{2.18}$$

Canonical paracontact connection on quasi-para Sasakian manifold is defined by

$$(\bar{\nabla}_X Y) = \nabla_X Y - g(X,\phi Y)\xi - \eta(Y)\phi X + \eta(X)\phi Y. \tag{2.19}$$

3 Curvature Tensor of Quasi-para-Sasakian Manifold with respect to canonical paracontact Connection

The curvature tensor $\bar{R}$ of Riemannian manifold M^n with respect to canonical paracontact connection $\bar{\nabla}$ is given by:

$$\bar{R}(X,Y)Z = \bar{\nabla}_X\bar{\nabla}_Y Z - \bar{\nabla}_Y\bar{\nabla}_X Z - \bar{\nabla}_{[X,Y]}Z. \tag{3.1}$$

In the view of equation (2.19), we have

$$\begin{aligned}\bar{R}(X,Y)Z = R(X,Y)Z &+ g(X,\phi Z)\phi Y - g(Y,\phi Z)\phi X \\ &- 2g(X,\phi Y)\phi Z + \{g(Y,Z)\eta(X) - g(X,Z)\eta(Y)\}\xi \\ &+ \{\eta(Y)X - \eta(X)Y\}\eta(Z),\end{aligned} \tag{3.2}$$

where

$$R(X,Y)Z = \nabla_X\nabla_Y Z - \nabla_Y\nabla_X Z - \nabla_{[X,Y]}Z \tag{3.3}$$

is the Riemannian cuevature tensor of Levi-Civita connection ∇.

Equation (3.2) is the relation between Riemannian curvature tensor with respect to canonical paracontact connection $\bar{\nabla}$ and Levita-Civita connection ∇.

Transvection of V in equation (3.2) gives

$$\begin{aligned}\bar{R}(X,Y,Z,V) = &R(X,Y,Z,V) + g(X,\phi Z)g(V,\phi Y) - g(Y,\phi Z)g(V,\phi X) \\ &- 2g(X,\phi Y)g(V,\phi Z) + \{g(Y,Z)\eta(X) - g(X,Z)\eta(Y)\}\eta(V) \\ &+ \{\eta(Y)g(X,V) - \eta(X)g(V,Y)\}\eta(Z),\end{aligned} \tag{3.4}$$

where

$$\bar{R}(X,Y,Z,V) = g(\bar{R}(X,Y)Z,V)$$

and

$$R(X,Y,Z,V) = g(R(X,Y)Z,V).$$

Putting $X = V = e_i$ $(1 \le i \le n)$ in equation (3.4) and taking summation ovre i, we get

$$\bar{S}(Y,Z) = S(Y,Z) + 2g(Y,Z) + (n-3)\eta(Y)\eta(Z). \tag{3.5}$$

where $\bar{S}$ and S denotes the Ricci tensor with respect to the connection $\bar{\nabla}$ and ∇ respectively.

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Again putting $Y = Z = e_i$ in equation (3.5) and taking summation over i, $(1 \le i \le n)$, we get

$$\bar{r} = r + 3n - 3, \tag{3.6}$$

where $\bar{r}$ and r denotes the scalar curvature with respect to the connection $\bar{\nabla}$ and ∇ respectively.

From equation (3.5), we have

$$\bar{Q}Y = QY + 2Y + (n-3)\eta(Y)\xi, \tag{3.7}$$

where $\bar{Q}$ and Q denotes the Ricci operator with respect to the connection $\bar{\nabla}$ and ∇ respectively and

$$\bar{S}(Y,\xi) = 0 = \bar{S}(\xi, Z). \tag{3.8}$$

Now from equation (3.2), we have

$$\bar{R}(X,Y)\xi = 0, \bar{R}(Y,\xi)Z) = 0, \bar{R}(\xi, Y)Z = 0. \tag{3.9}$$

Writting two more equation by the cyclic permutation of X,Y and Z in equation (3.2), we get

$$\begin{aligned}\bar{R}(Y,Z)X = {} & R(Y,Z)X + g(Y,\phi X)\phi Z - g(Z,\phi X)\phi Y - 2g(Y,\phi Z)\phi X \\ & + \{g(Z,X)\eta(Y) - g(Y,X)\eta(Z)\}\xi + \{\eta(Z)Y - \eta(Y)Z\}\eta(X)\end{aligned} \tag{3.10}$$

and

$$\begin{aligned}\bar{R}(Z,X)Y = {} & R(Z,X)Y + g(Z,\phi Y)\phi X - g(X,\phi Y)\phi Z - 2g(Z,\phi X)\phi Y \\ & + \{g(X,Y)\eta(Z) - g(Z,Y)\eta(X)\}\xi + \{\eta(X)Z - \eta(Z)X\}\eta(Y).\end{aligned} \tag{3.11}$$

Adding equations (3.2), (3.10) and (3.11) and using Bianchi's first identity, we have

$$\bar{R}(X,Y)Z + \bar{R}(Y,Z)X + \bar{R}(Z,X)Y = 4g(X,\phi Z)\phi Y. \tag{3.12}$$

Thus, we can state as follows

Theorem 3.1. *A Qusi para-Sasakian manifold M^n with canonical paracontact connection satisfies the equation (3.12).*

Theorem 3.2. *The curvature tensor of a canonical paracontact connection in a quasi para Sasakian manifalds is*

1. Skew-symmetric in first two slots.
2. Skew-symmetric in last two slots.
3. Symmetric in pair of slots.

Proof. 1. Interchanging X and Y in equation (3.4), we get

$$\begin{aligned}\bar{R}(Y,X,Z,V) = {} & R(Y,X,Z,V) + g(Y,\phi Z)g(V,\phi X) - g(X,\phi Z)g(V,\phi Y) \\ & - 2g(Y,\phi X)g(V,\phi Z) + \{g(X,Z)\eta(Y) - g(Y,Z)\eta(X)\}\eta(V) \\ & + \{\eta(X)g(Y,V) - \eta(Y)g(V,X)\}\eta(Z).\end{aligned} \tag{3.13}$$

Adding equations (3.4) and (3.13) with the fact that $R(X,Y,Z,V)+R(Y,X,Z,V) = 0$, we get

$$\bar{R}(X,Y,Z,V) + \bar{R}(Y,X,Z,V) = 0. \tag{3.14}$$

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